Calculus 1 Notes (2023/2024)

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1 Preliminaries

1.1 Trigonometry

Definitions

$$\tan x = \frac{\sin x}{\cos x} \qquad \sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x} \qquad \cot x = \frac{\cos x}{\sin x}$$
$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

Identities

$$\sin^2 x + \cos^2 x = 1$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\sin(2x) = 2\sin x\cos x$$
$$\cos(2x) = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$$

1.2 Absolute values

Definition: The **absolute value** is a number's distance from 0.

$$\begin{cases} |x| = x \text{ if } x \geq 0 \\ |x| = -x \text{ if } x < 0 \end{cases}$$

$$|ab| = |a||b|$$
$$a \leq |a|$$

Triangle inequality:

$$|a+b| \leq |a| + |b|$$

1.3 Complex numbers

i	imaginary number
$\|z\|$ / r	absolute value / modulus
$\overline{z}$	conjugate
θ	argument

$$i^2 = -1 \qquad z = a + bi = re^{i\theta} = r(\cos(\theta) + i\ \sin(\theta))$$

The **absolute value** or **modulus** of a complex number $z = a + bi$ is defined as: $|z| = \sqrt{a^2+b^2}$
Complex numbers follow the triangle inequality.
The **conjugate** of a complex number $z = a + bi$ is defined as $\overline{z} = a - ib$
Re z is the real part of a complex number z, Im z is the imaginary part.

Euler's Formula:

$$e^{i\theta} = \cos(\theta) + i\ \sin(\theta)$$

De Moivre's Formula:

$$z^n = r^n e^{in\theta}$$

2 Limits

2.1 Definition

A function f has a **limit** L if for every $\epsilon > 0$ there is a $\delta > 0$ such that
if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$
When proving using ϵ/δ, repeat the definition at the end of the proof.

A function can have at most 1 limit at any given point.
f has a **fixed point** at $a \in \text{dom}(f)$ if $f(a) = a$

$[a, b]$ is a closed interval, which includes endpoints.
(a, b) is an open interval, which doesn't include endpoints.

2.2 Limit theorems

The **intermediate value theorem** states that if f is a continuous function whose domain contains the interval [a, b], then it takes on any given value between $f(a)$ and $f(b)$ at some point within the interval.

If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$.

The **Squeeze Theorem** states that if $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$.

2.3 Limit Laws

Limit Laws can only be used if the limit exists.

1. **Sum Law**: The limit of a sum is the sum of the limits.
2. **Difference Law**: 2. The limit of a difference is the difference of the limits.
3. **Constant Multiple Law**: The limit of a constant times a function is the constant times the limit of the function.
4. **Product Law**: The limit of a product is the product of the limits.
5. **Quotient Law**: The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0).
6. **Power Law**: $\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n$ where n is a positive integer.
7. **Root Law**: $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$
8. $\lim_{x \to a} c = c$
9. $\lim_{x \to a} x = a$
10. $\lim_{x \to a} x^n = a^n$ where n is a positive integer.
11. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer.
    (If n is even, we assume that a > 0.)

2.4 Direct substitution

Direct Substitution Property:
If f is a polynomial or a rational function and a is in the domain of f , then:

$$\lim_{x \to a} f(x) = f(a)$$

When a is not in the domain of f we can use two different methods to calculate the limits with preliminary algebra, **factoring** and **conjugates**.
Both methods have the goal of eliminating the part of the equation that makes the limit impossible to calculate through direct substitution.

Example of calculating a limit by factoring:

$$\lim_{x \to 3} \frac{x^2 - 3x}{x^2 - x - 6} = \lim_{x \to 3} \frac{x(x-3)}{(x-3)(x+2)} = \lim_{x \to 3} \frac{x}{(x+2)} = \frac{3}{3+2} = \frac{3}{5}$$

The conjugate of $a + b$ is $a - b$. Multiplying $a + b$ with its conjugate gives $a^2 - b^2$.
Some fractions with $a + b$ as the numerator or denominator can by simplified by multiplying both sides of the fraction with $a - b$.
Conjugates can be used to eliminate square roots or absolute values.

2.5 Some nice limits

$$\lim_{h \to 0} \frac{\sin(h)}{h} = 1 \qquad \lim_{h \to 0} \frac{\cos(h) + 1}{h} = 0$$

2.6 L'Hôpital's rule

Also known as l'Hospital's rule.
If $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$ or $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = \pm\infty$ then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

3 Derivatives

3.1 Definition

The **derivative** of a real function f at $a \in \text{dom}(f)$ is:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \qquad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Definition of e:

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

f has a derivative at $a \in \mathbb{R}$ iff $f(x) = f(a) + Ax + E(x)(x - a)$ so that $A \in \mathbb{R}$ and $\lim_{x \to a} E(x) = 0$

3.2 Rules of differentiation

Suppose that f and g are functions such that $a \in \text{dom}(x) \cap \text{dom}(g)$

Sum Rule

$$(f+g)'(a) = f'(a) + g'(a)$$

Difference Rule

$$(f-g)'(a) = f'(a) - g'(a)$$

Constant Multiple Rule

$$(c * f)'(a) = c * f'(a)$$

Product Rule

$$(f * g)'(a) = f'(a) * g(a) + f(a) * g'(a)$$

Quotient Rule

$$(\frac{f}{g})'(a) = \frac{f'(a) * g(a) - f(a) * g'(a)}{[g'(a)]^2}$$

Power Rule

$$\frac{d}{dx}(x^n) = nx^{n-1} \text{ where } n \text{ is a positive integer}$$

Generalized Power Rule

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Chain Rule

Assume that $g(a) \in \text{dom}(f)$ and $f'(g(a))$ exists

$$[f(g(x))]'(x) = f'(g(x)) * g'(x)$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Basic Derivatives

$$(c)' = 0 \qquad (x)' = 1$$

Exponential functions

$$(e^x)' = e^x \qquad (a^x)' = a^x * \ln a$$

Logarithmic functions

$$(\ln(x))' = \frac{1}{x} \qquad (\log_a(x))' = \frac{1}{x \ln b}$$

Trigonometric functions

$$(\sin x)' = \cos x \qquad (\csc x)' = -\csc x \cot x \qquad (\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}} \qquad (\csc^{-1} x)' = -\frac{1}{x\sqrt{x^2-1}}$$

$$(\cos x)' = -\sin x \qquad (\sec x)' = \sec x \tan x \qquad (\cos^{-1} x)' = -\frac{1}{\sqrt{1-x^2}} \qquad (\sec^{-1} x)' = \frac{1}{x\sqrt{x^2-1}}$$

$$(\tan x)' = \sec^2 x \qquad (\cot x)' = -\csc^2 x \qquad (\tan^{-1} x)' = \frac{1}{1+x^2} \qquad (\cot^{-1} x)' = -\frac{1}{1+x^2}$$

3.3 Applications

The **tangent line** of $f(x)$ at $x = a$ has the formula

$$y = f'(a)(x-a) + f(a)$$

Given point (a, b):

$$y = f'(a)(x-a) + b$$

c is a **critical number** (local min/max) of f if $f'(c) = 0$ or $f'(c)$ does not exist.

Closed Interval Method:

To find the absolute max/min of f over $[a, b]$ we need to compare $f(c)$ for c critical, $f(a)$ and $f(b)$.

3.4 Mean Value Theorems

Rolle's Theorem
If f is continuous in $[a, b]$, is differentiable in (a, b) and $f(a) = f(b)$,
then there exists a $c \in (a, b)$ $(a < c < b)$ such that $f'(c) = 0$.
Fermat's Theorem
If f has a local min/max at c and $f'(c)$ exists then $f'(c) = 0$.
Mean Value Theorem
If f is continuous in $[a, b]$ and is differentiable in (a, b) then there exists a $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \text{ or, equivalently } f(b) - f(a) = f'(c)(b - a)$$

Cauchy's Mean Value Theorem
A curve that is not a function can be defined by $(f(t), g(t)), a \leq t \leq b$ where $f(t)$ and $g(t)$ are the values of x and y respectively. If f and g are continuous in $[a, b]$ and differentiable in (a, b) then there is a $c \in (a, b)$ such that

$$f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a))$$

If $g'(c) \neq 0$ and $g(b) - g(a) \neq 0$, then

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

3.5 Methods of differentiation

Implicit differentiation:

1. Take $\frac{d}{dx}$ of both sides, use the chain rule and the fact that y is a function of x.
2. Solve the resulting equation for y'
3. Substitute the values of x and y.

Logarithmic differentiation:

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the Laws of Logarithms to expand the expression.
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y' and replace y by $f(x)$.

4 Series

If $a_0, a_1, ...a_n$ are numbers and k is a dummy variable, then

$$\sum_{k=0}^{n} a_k = a_0 + a_1 + ... + a_n \qquad \prod_{k=0}^{n} a_k = a_0 * a_1 * ... * a_n$$

Sums of some finite series:

$$\sum_{k=1}^{n} n = \frac{n(n+1)}{2} \qquad \sum_{k=1}^{n} n^2 = \frac{n(n+1)(n+2)}{6}$$

The sum of the **infinite series** $\sum_{i=0}^{\infty} a_i$ is defined as

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=0}^{n} a_i$$

Sum of a $r \in \mathbb{R}$ **geometric series**

$$\sum_{k=0}^{n} r^k = \frac{1-r^n}{1-r} \qquad \text{If } |r| < 1, \text{ then } \sum_{i=0}^{\infty} r^k = \frac{1}{1-r}$$

Binomial Theorem if $n \in \mathbb{N}$

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \qquad \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Binomial Series if $n \in \mathbb{R}$

$$(1+x)^n = \sum_{k=0}^{\infty} \left[\frac{x^k}{k!} \prod_{i=0}^{k-1} (n-i) \right]$$

4.1 Taylor and Maclaurin Series

Taylor Series: $f(x) = \sum_{k=0}^{\infty} \frac{f^{[n]}(a)}{k!}(x-a)^k$ **Maclaurin Series**: $f(x) = \sum_{k=0}^{\infty} \frac{f^{[n]}(0)}{k!} x^k$

To calculate the n-th derivative of $f(x)$, use the Taylor series and the fact that $c_n = \frac{f^{[n]}(a)}{n!}$
Taylor Series can be used to calculate limits using the fact that the limit of a function is equal to the limit of its Taylor series.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

The Taylor series of $\ln(x)$ is centered at 1, the rest are centered at 0.

4.2 Convergent and divergent

If $\sum_{n=0}^{N-1} a_n$ has a limit as $N \to \infty$, then we say that the infinite series $\sum_{n=0}^{\infty} a_n$ is **convergent**. Otherwise, the series is **divergent**. If $\sum_{n=0}^{N-1} |a_n|$ has a limit as $N \to \infty$, then we say that the infinite series $\sum_{n=0}^{\infty} a_n$ is **absolutely convergent**.
To prove that a series converges, provide an upper bound that also converges.
To prove that a series diverges, provide a lower bound that also diverges.
Ratio Test Let $S = \sum_{n=0}^{\infty} a_n$ and $L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$. Then

- $L < 1$ implies that S is absolutely convergent.
- $L > 1$ implies that S is divergent.
- $L = 1$ implies nothing, the test is inconclusive.

Power series: $\sum_{n=0}^{\infty} c_n x^n$
A power series is absolutely convergent if $|x - a| < \lim n \to \infty \frac{|C_n}{|C_{n+1}|}$,
where the right side is the **radius of convergence**.

5 Integrals

5.1 Antiderivatives

$F(x)$ is the **antiderivative** of $f(x)$ if $F'(x) = f(x)$
If $F(x)$ has a zero derivative over an interval I then $F(x)$ is constant for $x \in I$.
If $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ then there is a constant C such that $F(x) = G(x) + C$.

Function	Antiderivative	Function	Antiderivative
$cf(x)$	$cF(x)$	$\sin x$	$-\cos x$
$f(x) + g(x)$	$F(x) + G(x)$	$\sec^2 x$	$\tan x$
$x^n (n \neq 1)$	$\frac{x^n + 1}{n + 1}$	$\sec x \tan x$	$\sec x$
$\frac{1}{x}$	$\ln \lvert x \rvert$	$\frac{1}{\sqrt{1 - x^2}}$	$\sin^{-1} x$
e^x	e^x	$\frac{1}{1 + x^2}$	$\tan^{-1} x$
b^x	$\frac{b^x}{\ln b}$	$\cosh x$	$\sinh x$
$\cos x$	$\sin x$	$\sinh x$	$\cosh x$

5.2 Methods of integration

Substitution Rule: If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

To replace the dx with du, you can use $dx = \dfrac{dx}{du}du$
When the integral contains $\sqrt[n]{ax+b}$, use $u = \sqrt[n]{ax+b}$
If $ad - bc \neq 0$, you can substitute $\sqrt[n]{\dfrac{ax+b}{cx+d}}$

Integration by Parts

$$\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int g(x)\,f'(x)\,dx$$

where g can be simplified by taking its derivative.
Let $u = f(x)$ and $v = g(x)$. Then, by the Substitution Rule:

$$\int u\,dv = uv - \int v\,du$$

Repeated integration by parts can lead a set of terms containing the integral, which can be set equal to the integral and this equation can be solved for the integral.

Trigonometric integrals
When dealing with odd powers of trig functions, use the substitution method, separate one of them and use $\sin^2(x) + \cos^2(x) = 1$.
When dealing with even powers of trig functions, use trig identities to rewrite the function as a sum of trig functions without exponents.
With a combination of even and odd powers, use the method for odd powers.
Weierstrass substitution for integrating rational trig functions:

$$u = \tan\left(\frac{x}{2}\right) \qquad dx = \frac{2}{1+u^2}\,du \qquad \sin x = \frac{2u}{1+u} \qquad \cos x = \frac{1-u^2}{1+u^2}$$

5.3 Partial fraction decomposition

$$\int \frac{P(u)}{Q(u)} \text{ where } P \text{ and } Q \text{ are polynomials of degree } m \text{ and } n \text{ respectively}$$

If $m \geq n$ use long division:

$$\frac{P(u)}{Q(u)} = p(u) + \frac{P^*(u)}{Q(u)} \text{ where } m^* < n$$

If $m < n$ then we need to find the roots of Q and use them in the PFD.
Case 1: If Q has n distinct real roots $u_1, u_2, \ldots, u_n$

$$\frac{P(u)}{Q(u)} = \frac{P(u)}{b_0(u-u_1)(u-u_2)\ldots(u-u_n)} = \frac{A_1}{u-u_1} + \frac{A_2}{u-u_2} + \ldots + \frac{A_n}{u-u_n}$$

where $A_1, A_2, \ldots, A_n$ are found by

1. multiplying both sides of the equation by the common denominator, factoring out the powers of u and solving a system of linear equations
2. **Heaviside's cover-up method**: take $u = a$ where a is a root in the fraction containing A_k, cover up $u - a$ on the left side, then substitute $u = a$, A_k = right hand side
3. using derivatives: $A_k = \dfrac{P(u_k)}{Q'(u_k)}$

Case 2: Q has repeated roots
Roots of Q: $u_1, u_2, \ldots, u_r$
Multiplicity (amount of repetitions of a root): $\alpha_1, \alpha_2, \ldots, \alpha_r$ where the sum of all α is n

$$\frac{P(u)}{Q(u)} = \frac{P(u)}{b_0(u-u_1)^{\alpha_1}(u-u_2)^{\alpha_2}\ldots(u-u_r)^{\alpha_r}} = \text{I'm not writing this in \LaTeX}$$

Special case: when $r = 1$ and $\alpha_1 = n$

$$\frac{P(u)}{Q(u)} = \frac{P(u)}{b_0(u-u_1)^n} = \frac{A_{11}}{u-u_1} + \frac{A_{12}}{(u-u_1)^2} + \ldots + \frac{A_{1n}}{(u-u_1)^n}$$

which is the Taylor polynomial of $P(u)$ centered at u_1

$$A_{1k} = \frac{P^{(n-k)}(u_1)}{b_0(n-k)!}$$

Case 3: Not all roots of Q are real numbers
Every polynomial can be factorized into linear and unfactorizable quadratic roots
where the fractions for the quadratics are of the form $\frac{B_k u + C_k}{u^2 + b_k u + c_k}$ These can be integrated by completing the square and using $[\arctan x]' = \frac{1}{1+x^2}$

5.4 Definite Integrals

The **definite integral** (Riemann sum) of $f(x)$ over $[a, b]$ is

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \qquad \text{Notation: } \int_b^a f(x)dx$$

$$\text{where } x_i^* \in [x_{i-1}, x_i],\ a = x_0 < x_1 < x_2 < \ldots < x_n = b$$

$$\text{and } \Delta x = \frac{b-a}{n},\ x_i = a + i\Delta x = a + \frac{i}{n}(b-a),\ i = 1, \ldots, n$$

Fundamental Theorem of Calculus: If $f(x)$ is continuous over $[a, b]$, then

$$g(x) = \int_a^x f(t)dt \text{ is continuous over } [a,b]\text{, differentiable over } (a,b)\text{, and } g'(x) = f(x)$$

$$\text{If } F(x) \text{ is an antiderivative of } f(x)\text{, then } \int_b^a f(x)dx = F(a) - F(b)$$

Properties of indefinite integrals

$$\int_b^a f(x)dx = -\int_a^b f(x)dx \qquad \int_a^a f(x)dx = 0 \qquad \int_a^b c * f(x)dx = c * \int_a^b f(x)dx$$

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx \qquad \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$\text{If } f(x) \geq 0 \text{ for } a \leq x \leq b\text{, then } \int_a^b f(x)dx \geq 0$$

$$\text{If } f(x) \geq g(x) \text{ for } a \leq x \leq b\text{, then } \int_a^b f(x)dx \geq \int_a^b g(x)dx$$

$$\text{If } m \leq f(x) \leq M \text{ for } a \leq x \leq b\text{, then } m(b-a) = \int_a^b m\,dx \leq \int_a^b f(x)dx \leq \int_a^b M\,dx = M(b-a)$$

5.5 Applications of integrals

Area between curves $f(x)$ and $g(x)$ with $g(x)$ on top $= \int_a^b [g(x) - f(x)]dx$

Average value of a function $f(x) = \dfrac{1}{b-a} \int_a^b f(x)dx$

Improper integrals: $\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$

Methods of **numerical integration** approximating $\int_a^b f(x)dx$

1. Integrating the Taylor polynomial up to n
   with error $\dfrac{f^{(n+1)}(d)}{(n+1)!}(x-c)$ where d is between x and c.
2. Midpoint Rule:
   using the Riemann sum where we take the value of f at the midpoint of each interval
   with error $|E_M| \leq \dfrac{K(b-a)}{24n^2}$
3. Trapezoid Rule:
   similar to the Riemann sum, but using trapezoids instead of rectangles
   with error $|E_T| \leq \dfrac{K(b-a)}{12n^2}$
4. Simpson's Rule:
   fitting arcs of parabolas on every set of 2 intervals

Comparison Test: If $f(x) \leq g(x)$ on an interval $[a, b]$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$

Arc length of a function on interval (a, b) can be approximated with a piecewise linear function
Exact arc length:

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} L_i = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

Volume of a solid of revolution:

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} \pi[f(x_i^*)]^2 \Delta x_i = \int_a^b \pi[f(x)]^2 dx$$

Surface area of a solid of revolution:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} A_i = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx$$

Probability Density Functions always have the properties $0 \leq f(x)$ and $\int_a^b f(x)\, dx = 1$

6 Ordinary Differential Equations

A **differential equation** is an equation containing a function and its derivatives as unknowns.
Ordinary means that the function has only one variable.
The variable x is the **independent variable**, the function y is the **dependent variable**.
The solution to an ODE can be multiplied by any constant C.
This constant C can be set by an **initial condition** which is some point (x, y) on the function.
An **ansatz** is an educated guess for the solution of an ODE.

6.1 First-order ODEs

Logistic equations stop increasing at a **carrying capacity** M:
$P'(t) = kP(t)\left[1 - \frac{P(t)}{M}\right]$ $\qquad P(t) = \dfrac{M}{\frac{M - P(0)}{P(0)}e^{-kt} + 1}$

Solving a **first-order linear ODE** $y'(x) + P(x)y(x) = Q(x)$:

1. If $Q(x) = 0$, then the equation is **homogeneous**.

$$y' + Py = 0 \qquad \frac{y'}{y} = -P \qquad \ln|y| = -\int P(x)\,dx \qquad y(x) = \pm e^{-\int P(x)\,dx}$$

2. If $Q(x) \neq 0$, then the equation is **inhomogeneous**.

$$I(x) = e^{\int P(x)\,dx} \qquad I(x)y'(x) + I(x)P(x)y(x) = I(x)Q(x)$$

where the left hand side is the derivative of a product. This leads to:

$$y(x) = e^{-\int P(x)\,dx}\int Q(x)e^{\int P(x)\,dx}\,dx$$

where both integrals in the exponents have the same C

6.2 Second-order ODEs

Systems of coupled ODEs can be represented using a vector $\boldsymbol{v}$ of functions: $\frac{d}{dt}\boldsymbol{v} = M\boldsymbol{v}$
A system can be **decoupled** using substitution to reach a solvable 2nd-order ODE.
The set of solutions of the linear ODE $y''(x) + y(x) = 0$ is a linear (vector) space.
The general solution of a 2nd-order ODE is a linear combination of functions.
Example: $y''(x) = y(x)$ $\qquad y(x) = C_1 \cos x + C_2 \sin x$
The solution set of an nth-order ODE is of dimension n.
2nd-order ODEs can be solved using e^{rx} as a guess and Euler's Formula.

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i} \qquad \sinh(ix) = i \sin x \qquad \cosh(ix) = \cos x$$

Solution of a homogeneous 2nd-order ODE with constant coefficients $ay''(x) + b'(x) + cy(x) = 0$

$$y(x) = e^{rx} \qquad ar^2 + br + c = 0 \text{ (\textbf{auxiliary equation})}$$

In the quadratic formula, if $\Delta = \sqrt{b^2 - 4ac} < 0$, there are complex solutions.

1. $\Delta > 0$, $r_1, r_2 \in \mathbb{R}$, $r_1 \neq r_2$
$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$
2. $\Delta = 0$, $r_1, r_2 \in \mathbb{R}$, $r_1 = r_2$
$$y(x) = C_1 e^{r_1 x} + C_2 x e^{r_2 x}$$
3. $\Delta < 0$, $r_1, r_2 \in \mathbb{C}$, $r_1 = \overline{r_2} = \alpha + \beta i$, $\alpha < 0$, $\beta > 0$
$$y(x) = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$$

Inhomogeneous case: $ay''(x) + b'(x) + cy(x) = G(x)$

1. Solve the homogeneous equation $ay''(x)+b'(x)+cy(x) = 0$, where $y_p(x)$ is the general solution.
2. Try to find a particular solution $y_p(x)$ of the inhomogeneous equation:
   (a) Guess some general $y_p(x)$ in the same form as $G(x)$.
   If $G(x)$ contains a polynomial, $y_p(x)$ has the same degree.
   If $G(x)$ contains a trig function, $y_p(x)$ is a linear combination of sines and cosines.
   (b) Substitute $y_p(x)$ and its derivatives into the ODE
   (c) Solve for constants using a system of linear equations.
3. Add the two solutions: $y(x) = y_c(x) + y_p(x)$
4. Match initial values.